ISSN 1028-3358, Doklady Physics, 2015, Vol. 60, No. 5, pp. 224–228. © Pleiades Publishing, Ltd., 2015.
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MECHANICS

Hierarchy of Nonlinear Models of the Hydrodynamics of Long Surface Waves

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Received December 11, 2014

DOI: 10.1134/S1028335815050079

For numerical solution of problems of hydrodynamics of long surface waves propagating in real reservoirs, it is advantageous to use models in which the region of applicability is associated with the characteristic scales of the wave process [1, 2]. An expansion of the circle of actual problems related to the development of coastal territories stimulates the creation of new models, and this fact results in variety of the systems of differential equations corresponding to these models due to a variety of methods and simplifying assumptions. Among these differential equations, there are also some that formally approximate the initial problem without providing an adequate description of the physical process or are inconvenient for numerical implementation.

The purpose of this study is formation of a uniform approach for constructing long-wave approximations in order to provide a hierarchical chain of shallow-water equations of the first and second approximations having a succession of physically substantial properties. Here we continue the recent investigations in [3–6]. It is necessary to acknowledge [7–10] and others as the first studies on this theme.

The derivation of shallow-water models taking into account the dispersion is based on the Euler equations for an ideal incompressible fluid on a rotating sphere, the mobility of the bottom surface being taken into account, while further passage along the hierarchy from completely nonlinear equations with dispersion towards simplifications proceeds with inheritance of the most important properties, in particular, the laws of conservation. We succeeded in writing the obtained nonlinear-dispersive (NLD) equations both on the plane and on the sphere in a universal compact form, which structurally coincides with the set of gas-dynamic equations.

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EULER'S EQUATIONS IN THE THIN-LAYER APPROXIMATION

The spherical system of coordinates $O\lambda\theta r$ is used with the origin at the center of a sphere of radius R rotating with a constant velocity Ω. We designated the longitude by λ, and the addition to the latitude φ $\left(-\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right)$ we denoted by $\theta = \frac{\pi}{2} - \varphi$, while r is the radial coordinate. It is assumed that the water layer is limited from below by an impenetrable mobile bottom $r = R - h(\lambda, \theta, t)$ and from above by a free surface $r = R + \eta(\lambda, \theta, t)$. As external forces, we considered only the force of Newtonian attraction $\mathbf{g}$ directed to the center of a rotating sphere. Considering the thickness $H = \eta + h$ of the water layer as small in comparison with R, we assume that the value of $g = |\mathbf{g}|$ and the water density ρ are constant in the entire layer, $\rho \equiv 1$. The mathematical models of the long-wave hydrodynamics are derived here from Euler's equations written with singling out the radial direction:

$$\begin{aligned}(JU^1)_\lambda + (JU^2)_\theta + (JW)_r &= 0,\\ \mathbf{V}_t + (\mathbf{U}\cdot\nabla)\mathbf{V} + W\mathbf{V}_r + \nabla P &= \mathbf{S},\\ W_t + \mathbf{U}\cdot\nabla W + WW_r + P_r &= -g + S_3,\end{aligned} \quad (1)$$

where the vectors $\mathbf{U} = (U^1, U^2) = (\dot{\lambda}, \dot{\theta})$ and $\mathbf{V} = (V_1, V_2)$ are composed correspondingly from the contravariant and covariant components of the "horizontal" component of the velocity vector; in this case, $V_1 = (\Omega + U^1)r^2\sin^2\theta$, $V_2 = r^2U^2$, the radial velocity component is designated by $W = \dot{r}$, $J = -r^2\sin\theta$, $\mathbf{S} = (0, S_2)$, $S_2 = (\Omega + U^1)^2r^2\sin\theta\cos\theta$, $S_3 = (\Omega + U^1)^2r\sin^2\theta + (U^2)^2r$, and P is the pressure, $\nabla = \left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\theta}\right)$. On the layer boundaries, the boundary conditions are put

$$(\eta_t + \mathbf{U}\cdot\nabla\eta - W)\big|_{r = r_\eta} = 0,$$

$$P\big|_{r = r_\eta} = 0, \quad (h_t + \mathbf{U}\cdot\nabla h + W)\big|_{r = r_h} = 0,$$

where $r_\eta = R + \eta$, $r_h = R - h$.

The derivation of the shallow-water equations assumes singling out the basic scales. Let L and h_0 be the characteristic sizes in the "horizontal" and radial directions in the layer, a_0 is the characteristic amplitude of waves, while $\alpha = \frac{a_0}{h_0}$, $\mu = \frac{h_0}{L}$, and $\varepsilon = \frac{h_0}{R}$ are the parameters of nonlinearity, frequency dispersion, and relative thickness of the water layer, respectively. The dimensionless variables are defined from the relations

$$(\lambda', \theta') = \frac{(\lambda, \theta)}{\lambda_0}, \quad (h', r', H') = \frac{(h, r, H)}{h_0},$$

$$\eta' = \frac{\eta}{a_0}, \quad t' = \frac{t}{t_0}, \quad \Omega' = \frac{\Omega}{\omega_0},$$

$$(U^\beta)' = \frac{U^\beta}{\omega_0}, \quad (V_\beta)' = \frac{V_\beta}{R\sqrt{gh_0}} \quad (\beta = 1, 2),$$

$$W' = \frac{W}{\mu\sqrt{gh_0}}, \quad P' = \frac{P}{gh_0},$$

where $\lambda_0 = \frac{L}{R}$, $t_0 = \frac{L}{\sqrt{gh_0}}$, and $\omega_0 = \frac{\lambda_0}{t_0}$. Writing Euler's equations (1) in dimensionless variables and rejecting the terms of the order of $O(\varepsilon)$, we come [5] to the thin-layer approximation. Adding the dimensionless boundary conditions, we obtain a problem with small parameters α and μ, which is convenient for deriving the shallow-water models.

HIERARCHY OF MODELS IN THE SPHERICAL GEOMETRY

The hierarchical chain of the shallow-water models of the second hydrodynamical approximation has a model in its apex in which the desired values are the total depth H of the fluid and a certain vector function $\mathbf{u}_a$ approaching the "horizontal" velocity $\mathbf{U}$:

$$\mathbf{U}(\lambda, \theta, r, t) = \mathbf{u}_a(\lambda, \theta, t) + \mu^2 \mathbf{C}(\lambda, \theta, r, t). \qquad (2)$$

For example, the "horizontal" velocity of flow on a quite certain surface $r = r_a(\lambda, \theta, t)$ lying between the bottom and the free boundary, i.e., $\mathbf{u}_a(\lambda, \theta, t) = U(\lambda, \theta, r_a(\lambda, \theta, t), t)$ is accepted for $\mathbf{u}_a$ in [11]. After determining the choice of velocity $\mathbf{u}_a$, we obtain the set of branchings of hierarchical chains of the shallow-water models.

Let us consider one such chain, which is obtained when choosing $\mathbf{u}_a$ in the form of the "horizontal" component of the 3D-flow velocity averaged over the fluid-layer thickness:

$$\mathbf{u} = (u^1, u^2) = \frac{1}{H}\int_{r_h}^{r_\eta} \mathbf{U}\,dr. \qquad (3)$$

Using expansion (2) with $\mathbf{u}_a = \mathbf{u}$ and carrying out the transformations of equations in the thin-layer approximation with an accuracy $O(\mu^2)$ inclusively, we come to the closed NLD model describing the dynamics of long waves in the rotating spherical system of coordinates:

$$H_t + \nabla \cdot (H\mathbf{u}) = 0, \qquad (4)$$

$$\mathbf{v}_t + (\mathbf{u} \cdot \nabla)\mathbf{v} + \frac{\nabla p}{H} = \frac{\pi_0}{H}\nabla h + \mathbf{s}, \qquad (5)$$

where $\mathbf{v} = (v_1, v_2)$, $v_1 = (\Omega + u^1)R^2\sin^2\theta$, $v_2 = R^2u^2$, $\mathbf{s} = (0, s_2)$, $s_2 = (\Omega + u^1)^2R^2\sin\theta\cos\theta$,

$$p = \int_{r_h}^{r_\eta} \pi\,dr = \frac{gH^2}{2} - \frac{H^3}{3}q_1 - \frac{H^2}{2}q_2, \qquad (6)$$

$\pi(r)$ is the distribution of the main part of pressure P in the long-wave approximation on the coordinate r ($r_h \le r \le r_\eta$):

$$\pi(r) = g(H - (r - r_h)) - (H - (r - r_h))q_2 - \left(\frac{H^2}{2} - \frac{(r - r_h)^2}{2}\right)q_1, \qquad (7)$$

$$q_1 = D(\nabla \cdot \mathbf{u}) - (\nabla \cdot \mathbf{u})^2, \quad q_2 = D^2 h,$$

$$\pi_0 = \pi|_{r = r_h} = gH - \frac{H^2}{2}q_1 - Hq_2,$$

$$D = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \quad \nabla \cdot \mathbf{u} = \frac{(J_0u^1)_\lambda + (J_0u^2)_\theta}{J_0}, \qquad (8)$$

$$J_0 = -R^2\sin\theta.$$

The derivation of Eqs. (4), (5), uses a number of methods developed in [3–5]. It should be noted that no additional restrictions, except Eq. (2), were used in the derivation in contrast to, for example, [3, 11], where, in essence, the assumption of potentiality of the initial 3D flow is used for constructing the NLD models in the spherical geometry.

It is important that Eq. (5) of motion assumes writing the momentum-balance equation in quasi-conservative form:

$$(H\mathbf{v})_t + \nabla \cdot (H\mathbf{u} \otimes \mathbf{v}) + \nabla p = \pi_0 \nabla h + H\mathbf{s}, \qquad (9)$$

where $\mathbf{u} \otimes \mathbf{v}$ is the tensor product of vectors. Equations (4), (9) have the same structure as the gas-dynamics equations; therefore, it is possible to use similar methods for the numerical solution.

One more important property of NLD model (4), (5) is that it, in contrast to the NLD model [11], assumes the law of total-energy balance as a consequence consistent with a similar law for the 3D equations in the thin-layer approximation. For consistency, we mean the following. If we use the expansion of velocity with respect to the parameter μ^2 in the law of total-energy conservation for the 3D model and reject the terms of the order of $O(\mu^4)$, we come to the same equation of the total-energy balance, which follows directly from NLD Eqs. (4), (5) after the corresponding equivalent transformations:

$$(HE)_t + \nabla \cdot \left(\mathbf{u}H\left(E + \frac{p}{H}\right)\right) = -\pi_0 h_t. \quad (10)$$

In this case, the total energy of the 3D flow averaged over the layer thickness with taking into account representation (2) is accepted as the expression for the total energy E in the NLD model:

$$E = K + g\left(\frac{H}{2} - h\right) - \frac{\Omega^2 R^2 \sin^2\theta}{2}, \quad (11)$$

where the kinetic energy K is expressed through the variables of the NLD model from the formula

$$K = \frac{1}{H}\int_{r_h}^{r_\eta} \frac{\mathbf{u}\cdot\mathbf{u} + w^2}{2} dr$$
$$= \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{H}{2}(\nabla\cdot\mathbf{u})Dh + \frac{H^2}{6}(\nabla\cdot\mathbf{u})^2 + \frac{(Dh)^2}{2}, \quad (12)$$

$\mathbf{u}\cdot\mathbf{u} = (u^1 R\sin\theta)^2 + (u^2 R)^2$, $w = -Dh - (r - r_h)(\nabla\cdot\mathbf{u})$ (w is the main part of the radial velocity W in the expansion with respect to μ^2).

In the case of a motionless bottom, Eq. (10) accepts a conservative form, which completely coincides with the form of the law of total-energy conservation in an ideal gas. It should be noted that the presence of the consistent balance equation of total energy makes it possible to carry out an additional control of calculations using the numerical methods of the solution instead of only confirming the physical validity of the NLD model.

Further downwards along the hierarchical chain, in the shallow-water models following from total NLD model (4), (9) (obtained without the assumption of the smallness of parameter α) at various simplifying assumptions, the equation of continuity for all models has the same form of Eq. (4). If, for example, we assume that $\alpha = O(\mu^2)$ and use the equality $H' = h' + \alpha\eta'$ in the dimensionless expressions for the average pressure $\frac{p}{H}$ and the relative pressure $\frac{\pi_0}{H}$ to the bottom, we come to weakly dispersive models in the spherical geometry after rejecting the terms of the order of $O(\alpha\mu^2)$ and $O(\alpha^2\mu^2)$ and returning to dimensional values. The equation of motion of this model retains the form of Eq. (9), and only expressions (6), (8) for p and π_0 are changed:

$$p = \frac{gH^2}{2} - \frac{Hh}{3}D(h\nabla\cdot\mathbf{u}) - \frac{Hh}{2}D^2 h,$$
$$\pi_0 = gH - \frac{H}{2}D(h\nabla\cdot\mathbf{u}) - HD^2 h. \quad (13)$$

This model also admits the balance equation of total energy as a consequence consistent with the energy-balance law in the 3D model, which has the same form of Eq. (10) as in the total NLD model, only now the kinetic energy is calculated from the modified formula

$$K = \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{h}{2}(\nabla\cdot\mathbf{u})Dh + \frac{h^2}{6}(\nabla\cdot\mathbf{u})^2 + \frac{(Dh)^2}{2}. \quad (14)$$

Omitting the description of other links of the hierarchical chain under consideration, we pass to its lower link—the classical (dispersionless) shallow-water models on a sphere. It can be obtained, for example, if we neglect all dispersive terms, i.e., the terms of the order $O(\mu^2)$ in dimensionless expressions (13). Then we obtain again the same set of Eqs. (4), (9) but with other expressions for p and π_0:

$$p = \frac{gH^2}{2}, \quad \pi_0 = gH. \quad (15)$$

For the classical shallow-water model, the form of energy Eq. (10) also remains constant; however, the kinetic energy is calculated now from the simplified formula $K = \frac{\mathbf{u}\cdot\mathbf{u}}{2}$.

Thus, the hierarchical chain of nonlinear shallow-water models enclosed in each other in the spherical geometry based on the stage-by-stage simplification of the dispersive component is constructed, the form of writing of all models in the chain and the form of Eqs. (4), (9), (10), for the balance relations being retained.

HIERARCHY OF SHALLOW-WATER MODELS ON A PLANE

We consider one more vertical hierarchical chain of the shallow-water models, in the derivation of which the fluid flow is initially considered in the Cartesian system of coordinates $Oxyz$, the axis Oz of which is directed vertically upwards, and the coordinate plane Oxy coincides with the horizontal unperturbed free surface. In this case, we consider that the layer of an ideal incompressible fluid limited from below by a mobile bottom set by the function $z = -h(x, y, t)$ and from above by the free boundary $z = \eta(x, y, t)$ [1, 2].

Substituting the analogue of representation (2) in Euler's equations

$$\mathbf{U}(x, y, z, t) = \mathbf{u}_a(x, y, t) + \mu^2\mathbf{C}(x, y, z, t) \quad (16)$$

and choosing the velocity $\mathbf{u}_a$ in the approximate model from the formula

$$\mathbf{u} = (u, v) = \frac{1}{H}\int_{-h}^{\eta}\mathbf{U}dz, \quad (17)$$

similarly to Eq. (3), we come to a completely nonlinear NLD model on the plane, the equations of which have the same form of Eqs. (4), (5) as in the spherical coordinates with only difference being that in the plane case $\mathbf{s} = 0$, $\mathbf{v} = \mathbf{u}$, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, $\nabla\cdot\mathbf{u} = u_x + v_y$, and the analogue of the formula (7) is used for calculating the pressure in the NLD model:

$$\pi(z) = g(H-(z+h)) - (H-(z+h))q_2$$
$$-\left(\frac{H^2}{2} - \frac{(z+h)^2}{2}\right)q_1.$$

Momentum-balance equation (9) here takes the form

$$(H\mathbf{u})_t + \nabla\cdot(H\mathbf{u}\otimes\mathbf{u}) + \nabla p = \pi_0\nabla h, \quad (18)$$

and the balance equation of total energy accepts the same form of Eq. (10) as in the spherical case but, instead of expression (11) for the total energy, it is necessary now to take the formula

$$E = K + g\left(\frac{H}{2} - h\right) \quad (19)$$

and to replace $\mathbf{u}\cdot\mathbf{u}$ with $(u^2 + v^2)$ in expression (12) for the kinetic energy.

By analogy to the spherical case, we derive the weakly dispersive equations and the shallow-water equations of the first approximation, which is written in the form of Eqs. (4), (18). Like the set of total NLD equations on the plane, they are invariant with respect to Galileo's transform and have balance equation (10) of total energy consistent with the 3D model, the pressure being determined by formulas (13) or (15), respectively.

The models of this vertical chain are derived from the 3D Euler equations, which were written in the Cartesian system of coordinates. However, the same models on the plane can be obtained directly from the corresponding spherical models by establishing the "horizontal" relations due to the limiting transition from the spherical geometry to the plane one. We consider, for example, the total NLD model with Eqs. (4), (9) and expression (7) for the pressure. To obtain the analogue of this model on the plane, it is necessary first to redefine the functions $h(\lambda, \theta, t)$ and $\eta(\lambda, \theta, t)$ so that they set the deviations $h_0(\lambda, \theta, t)$, $\eta_0(\lambda, \theta, t)$ of the surfaces described by them from an unperturbed free boundary of the water layer instead of a sphere of radius R. For the shallow-water models under consideration in the spherical geometry, the unperturbed free boundary is described by the equation $r = R + z_0(\theta)$, where

$$z_0(\theta) = \frac{1}{2g}\Omega^2R^2\sin^2\theta + \text{const},$$

therefore, the following equalities hold

$$-h = z_0 - h_0, \quad \eta = z_0 + \eta_0,$$
$$D^2h = D^2h_0 - D^2z_0,$$

and the modified momentum-balance equation can be written in the former form of Eq. (9) with taking into account that now

$$s_2 = (2\Omega u^1 + (u^1)^2)R^2\sin\theta\cos\theta.$$

Further, in a certain vicinity of the fixed point (λ_*, θ_*), we consider the transformation of coordinates

$$x = R(\lambda - \lambda_*)\sin\theta_*, \quad y = -R(\theta - \theta_*)$$

and introduce the components of the velocity vector $\mathbf{u} = (u, v)$: $u = \dot{x} = Ru^1\sin\theta_*$, $v = \dot{y} = -Ru^2$. The circumpolar regions are not considered, i.e., $\theta_0 \le \theta \le \pi - \theta_0$, $\theta = \text{const} > 0$. Then it is possible to consider that the function $\sin^{-1}\theta$ is limited. We also assume that the variables H, u, v, and their derivatives are limited. If we assume that the region is small in the direction of latitude, i.e., the value of $\delta = \theta - \theta_*$ is small, passing to new variables in mass-balance equation (4) and in the modified momentum-balance equation, and neglecting the terms of the order of $O(\delta)$ or $O(1/R)$ in the obtained equations, we come to the set of equations of the total NLD model on the plane

$$H_t + \nabla\cdot(H\mathbf{u}) = 0, \quad (20)$$

$$(H\mathbf{u})_t + \nabla\cdot(H\mathbf{u}\otimes\mathbf{u}) + \nabla p = \pi_0\nabla h + H\mathbf{f}, \quad (21)$$

the momentum-balance equation of which differs from Eq. (18) by the presence of the term with the vector $\mathbf{f} = (f_1, f_2)$ reflecting the effect of the Coriolis force: $f_1 = f_* v$, $f_2 = -f_* u$, $f_* = 2\Omega\cos\theta_*$ is the constant Coriolis coefficient.

It should be noted that the balance equation of total energy for model (20), (21), coincides with that, which took place for the NLD model considered above on the plane obtained at $\Omega = 0$. It is clear that NLD model (20), (21) generates a new vertical hierarchical chain of the shallow-water models on the plane with taking into account the Coriolis force.

Thus, the hierarchical chains of the mathematical shallow-water models enclosed in each other are constructed on a rotating sphere and on a plane; they have the balance relations of identical structure similar to that of the gas-dynamics equations. It means that for the numerical solution of problems using the constructed NLD models, it is possible to apply the algorithms based on well investigated numerical methods of gas dynamics similar to how it is implemented within the framework of the classical dispersionless shallow-water equations, while the presence of the law of total-energy balance plays an important role for verification of the numerical algorithms and the control of calculations.

It is obvious that it is possible to expand the set of hierarchical chains considered here while adding new ones. For example, each nonlinear model naturally generates a chain of linearized shallow-water models enclosed in each other with the same form of writing the balance relations and also serves as the origin of the chain of models with decreasing dimension on space. Other chains are obtained if we take a value different from the average velocity (17) as the velocity of the approximate model. The models of such chains can have improved dispersive properties [12]; however, in the case of a mobile bottom, they have an extremely cumbersome form of writing the equations [13] and have no balance law of total energy. Therefore, a modification of the models of these chains is required. Certain steps in this direction are made in [14] in which, in particular, it was shown that the modified models of this group admit the writing of the momentum-balance equation in a laconic quasi-conservative form.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project no. 14-17-00219.

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Translated by V. Bukhanov