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# Characteristics of finite difference methods for dispersive shallow water equations

**Abstract:** The paper contains a description of the most important properties of numerical methods for solving nonlinear dispersive hydrodynamic equations and their distinctions from similar properties of finite difference schemes approximating classic dispersion-free shallow water equations.

**Keywords:** Nonlinear dispersive equations, finite difference methods, accuracy, stability, dispersion.

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Papers [19, 20] are commonly considered as a starting point of numerical simulation of surface waves within the framework of nonlinear-dispersive (NLD) models. These papers present a model NLD equation for a unidirectional propagation of waves in a channel (the so-called regularized equation of long waves or RLW-equation), a Boussinesq system of equations for a curvilinear bottom surface, and a finite difference scheme approximating those equations. The appearance of these papers initiated a sustained interest to derivation of various variants of NLD models and the corresponding algorithms of their numerical implementation.

The first finite difference methods were based on known difference schemes for classic shallow water equations, at that moment these equations were the most common model for studying long surface waves. These equations are a system of nonhomogeneous quasilinear equations of hyperbolic type. To solve these equations the theory and technique of the numerical implementations were developed to the middle of the seventieth (see, e.g., [21]).

Sometimes, constructing difference schemes for NLD equations, researches used quite simple techniques, ignored the theory, and applied a direct term-wise approximation of derivatives entering the differential equation. The papers focused on numerical modelling mainly discussed the formulation of a physical problem, test and experimental data, but the features of the corresponding numerical method and its testing remained outside publications. The issues that the results of numerical calculations could describe not only a physical phenomenon, but also reflect intrinsic properties of a difference scheme were barely discussed.

A comparative analysis and study of some properties of several finite difference schemes for a regularized equation was first given in [4]. In [2], a generalization of the Galerkin method forming the base of the finite element method was proposed for the same equation. Later, the method of finite differences was widely used for solution of NLD equations in comparison with the finite element method because the latter appeared to be more laborious and its main advantage, calculations in complicated domains, was implemented with lesser costs with the use of the finite volume method.

In this paper we focus on the method of finite differences because this is where we can demonstrate the features of numerical solution of NLD equations differing from classic dispersion-free shallow water equations by the presence of the third-order derivatives with mixed spatial and time derivatives which requires special approaches to construction of numerical algorithms.

The present study showed that stability conditions for difference schemes for NLD equations and also conditions of appropriate approximation of dispersion terms have specific features and automatic declaration of properties obtained for equations of the first approximation does not provide an adequate description of actual properties of the numerical model. For example, it was shown that in contrast to the stability conditions for the hyperbolic case specified as restrictions on the Courant number, in this case we get an additional

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parameter relating the spatial mesh size to a typical depth of the basin. Another example is the following: if in the case of hyperbolic equations the ‘scheme’ dispersion is clearly seen, then it can be removed, if necessary, by developed methods of monotonization of solutions; however, in the case of NLD equations we need a fine analysis of relations between the ‘scheme’ and ‘physical’ dispersions in order to provide a description of the actual dispersion pattern of the flow.

We can also indicate other features of the numerical implementation of NLD equations such as a variety of approaches to the splitting of NLD equations and respective differential schemes for construction of the efficient algorithms. In addition to known methods of splitting with respect to spatial directions (reduction to one-dimensional problems) and with respect to physical processes widely used in calculations of multi-dimensional systems of quasilinear equations, various analytical splitting methods were developed for NLD models. These methods can be sometimes interpreted as a change of variables allowing one, for example, to solve separately evolution equations and equations dependent only on spatial variables.

There are also a number of other problems inherent to numerical implementation of NLD models. For example, there is a problem of coordination of properties of a difference scheme and the method implementing it, which additionally may require certain restrictions on the parameters of the difference scheme and which may reduce the advantages of implicit schemes.

Thus, in spite of the existing hierarchy of hydrodynamic models [24] associated with the derivation of models and their properties and the accordingly generated hierarchy of numerical methods, the effective use of the higher-order approximation models requires consideration of both the succession of properties of difference schemes and the acquisition of new qualities.

# 1 Hierarchy of shallow water models

In spite of the variety of NLD models, they all can be combined in a form of a basic model containing a parameter-function whose specification can produce many known systems of NLD equations [7]. The techniques of construction of numerical algorithms are given below in their application to the basic NLD model. We restrict ourselves here with the study of the flow of an ideal incompressible fluid between the movable bottom $z = -h(x, t)$ and the free boundary $z = \eta(x, t)$ in the Cartesian coordinate system $Oxz$ whose axis $Oz$ is directed vertically and the axis $Ox$ coincides with the unperturbed water surface. NLD equations follow from the Euler equation under the assumption that the flow has a long-wave nature, i.e., the ratio $\mu = h_0/L$ is small and we can neglect the values of the order $O(\mu^4)$. Here $L$ and $h_0$ are typical dimensions in horizontal direction and depth, respectively. The required values are the total depth $H = h + \eta$ and a certain function $u(x, t)$ approximating the fluid velocity and interpreted as the velocity in the approximate model. Most often, $u(x, t)$ is taken as the horizontal component of the velocity vector $U = U(x, z, t)$ in the Euler equation on some surface $z = z_u(x, t)$ included into the liquid layer, or the mean water velocity relative to the thickness of the layer.

## 1.1 The basic model

In the derivation of the basic NLD model we assume that the following expansion is valid:

$$U(x, z, t) = u(x, t) + \mu^2 v(x, z, t). \tag{1.1}$$

Given condition (1.1) and retaining terms up to the order $O(\mu^2)$, a basic NLD model was obtained in [7] and its one-dimensional equations has the following form:

$$H_t + [H(u + J)]_x = 0, \quad u_t + uu_x + \frac{p_x}{H} = \frac{p_0}{H} h_x - \frac{1}{H}\Big[(HJ)_t + u(HJ)_x + 2HJu_x\Big] \tag{1.2}$$

where $p$ is the pressure of the NLD model integrated over the thickness of the layer, $p_0$ is the pressure on the bottom, i.e.,

$$p = g\frac{H^2}{2} - \frac{H^3}{3}(D(u_x) - u_x^2) - \frac{H^2}{2}D^2 h, \qquad p_0 = gH - \frac{H^2}{2}(D(u_x) - u_x^2) - HD^2 h \tag{1.3}$$

$$J = \frac{1}{H}\int_{-h}^{\eta} v\,\mathrm{d}z, \qquad D = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}.$$

The known NLD models can be obtained form basic model (1.2) under an appropriate choice of the function $J$ [7, 24].

## 1.2 The case $J = 0$

If in addition to condition (1.1) we suppose the horizontal component of the velocity vector $U$ does not depend on the vertical coordinate $z$, or $u(x, t)$ is taken as the velocity averaged over depth and defined by the formula

$$u = \frac{1}{H}\int_{-h}^{\eta} U(x, z, t)\,\mathrm{d}z$$

then $J \equiv 0$ and the equations of the basic model become simpler and take the form

$$H_t + (Hu)_x = 0, \quad (Hu)_t + (Hu^2 + p)_x = p_0 h_x \tag{1.4}$$

where $p$ and $p_0$ are calculated by formulas (1.3). The models of such class are, for example, the models of Green–Naghdi [5], Zheleznyak–Pelinovsky [27], Fedotova–Khakimzyanov [6, 8], and many others following from the indicated models under transformation of the form of dispersive terms of the motion equation with the use of the continuity equation. Other modifications used to simplify dispersive terms are also possible (for example, using restrictions on the bottom geometry and flow patterns [1, 20, 22]).

Analyzing difference schemes, it is often sufficient to consider only NLD equations describing flows over a horizontal bottom $h(x, t) \equiv h_0$. In this case equations (1.4) are simplified as

$$H_t + (Hu)_x = 0, \quad (Hu)_t + (Hu^2 + p)_x = 0 \tag{1.5}$$

where $H = h_0 + \eta$ and the dispersive component of the pressure has a cubic dependence on the total depth, i.e.,

$$p = g\frac{H^2}{2} - \frac{H^3}{3}(u_{xt} + uu_{xx} - u_x^2). \tag{1.6}$$

A modification of dispersive terms of NLD model (1.5), (1.6) was proposed in [8], which preserves the basic properties of the model and at the same time simplifies numerical implementation. For one-dimensional flows over a horizontal bottom these equations have the same form as (1.5) with the only difference that the function $p$ is determined by the following formula with linear dependence of the dispersive component of the pressure on the total depth:

$$p = g\frac{H^2}{2} - H\frac{h_0^2}{3}(u_{xt} + uu_{xx}). \tag{1.7}$$

Equations (1.5), (1.7) were derived in [8] from equations of complete NLD model (1.5), (1.6) written in dimensionless variables and truncating terms of the order $O(\alpha\mu^2)$ in the averaged pressure $p/H$, where $\alpha = a_0/h_0$ is the parameter of nonlinearity. The weakly nonlinear Peregrine model [20] can be also obtained from the complete model by rejecting terms of the same order of smallness, but now from the expression $p_x/H$, i.e.,

$$H_t + (Hu)_x = 0, \quad u_t + uu_x + g\eta_x = \frac{h_0^2}{3}u_{xxt}. \tag{1.8}$$

In contrast to NLD models (1.5), (1.6) and (1.5), (1.7), Peregrine model (1.8) does not possess the law of total energy conservation [8] and the equation for the total momentum $Hu$ cannot be written in conservative form (1.5).

Being different in the nonlinear case, after linearization all models [5, 6, 20, 27] take the same form and in the case of one-dimensional flows with zero background velocity their equations are

$$\eta_t + h_0 u_x = 0, \quad u_t + g\eta_x = \frac{h_0^2}{3} u_{xxt}. \tag{1.9}$$

Finite difference approximations of these linear equations are used to reveal necessary stability conditions for nonlinear difference schemes and also to study their dissipative and dispersive properties. If we suppose that waves spread in the same direction, then system of equations (1.9) imply the so-called scalar RLW equation [19]:

$$\eta_t + c_0\eta_x + \frac{3c_0}{2h_0}\eta\eta_x = \frac{h_0^2}{6}\eta_{xxt} \tag{1.10}$$

where $c_0 = \sqrt{gh_0}$, which together with its linear analogue closes the hierarchical chain of NLD models.

### 1.3 The case $J \neq 0$

Another class of known models is specified by the choice of the velocity $u$ on a certain surface $z = z_u(x, t)$. For example, NLD models from [18, 26] were derived for a stationary bottom with $z_u = -0.531h$. In the case when the original 2D-flow is potential the function $J$ can be specified explicitly (see [7]):

$$J = -\Big[\frac{H}{2} - (z_u + h)\Big](h_{xt} + 2u_x h_x + u h_{xx}) - \Big[\frac{H^2}{6} - \frac{(z_u + h)^2}{2}\Big]u_{xx}. \tag{1.11}$$

In the case of a fixed horizontal bottom the linear variant of the basic model for $z_u = \text{const}$ takes the form

$$\eta_t + h_0 u_x = -\Big(\beta + \frac{1}{3}\Big)h_0^3 u_{xxx}, \quad u_t + g\eta_x = -\beta h_0^2 u_{xxt} \tag{1.12}$$

where $\beta = z_u^2/(2h_0^2) + z_u/h_0$. For $\beta = -1/3$ relations (1.12) give equations (1.9). Besides, it is easy to calculate $\beta$ for the models from [18, 26].

### 1.4 Classic shallow water equations

Dispersion-free shallow water equations follow from the basic model if we assume $p = gH^2/2$, $p_0 = gH$, and $J = 0$. In the case of an even bottom their form coincides with (1.5) and after linearization they can be written in form (1.9) with removed dispersive summand:

$$\eta_t + h_0 u_x = 0, \quad u_t + g\eta_x = 0. \tag{1.13}$$

## 2 Comparative stability analysis of difference schemes for equations with dispersive and dispersion-free equations

One of the simplest way to construct a difference scheme for an NLD equation consists in direct approximation of its derivatives by finite differences. Such difference schemes were first studied in [4]. Consider them relative to RLW-equation (1.10) rewriting it in the form

$$\eta_t + c_0\eta_x + a\eta\eta_x = \nu\eta_{xxt} \tag{2.1}$$

where $a = 3c_0/2h_0$, $\nu = h_0^2/6$. The choice of this equation as a model equation is good not only because it describes nonlinear and dispersive processes, but also due to its thoroughly studied analytic properties. In particular, it was shown in [3] that it is well definite in the sense of Hadamard (this equation is also called BBM equation due to importance of this property), it has an analytic solution in the form of a solitary wave, and the asymptotic behaviour of short waves in linear approximation is the same as for the Peregrine NLD model.

Let $\Delta x$ and $\tau$ be the steps of a uniform grid for the variables $x$ and $t$, $\eta_j^n$ be the values of the grid function $\eta$ at grid nodes $(x_j,\ t^n)$. Writing finite difference schemes, we use the following notations for difference derivatives:

$$\eta_{x,j}^n = \frac{\eta_{j+1}^n - \eta_j^n}{\Delta x}, \qquad \eta_{\bar{x},j}^n = \frac{\eta_j^n - \eta_{j-1}^n}{\Delta x}, \qquad \eta_{\mathring{x},j}^n = \frac{\eta_{j+1}^n - \eta_{j-1}^n}{2\Delta x}$$

$$\eta_{t,j}^n = \frac{\eta_j^{n+1} - \eta_j^n}{\tau}, \qquad \eta_{\bar{x}x,j}^n = \frac{\eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n}{\Delta x^2}, \qquad \eta_{\bar{x}xt,j}^n = \frac{\eta_{\bar{x}x,j}^{n+1} - \eta_{\bar{x}x,j}^n}{\tau}.$$

The first of difference schemes studied in [4] was proposed previously in [19]. Using the above notations and omitting the subscript $j$, we write it down in the following form:

$$\eta_t^n + (c_0 + a\eta^n)\Big(\frac{1}{2}\eta_{\mathring{x}}^{n+1} + \frac{1}{2}\eta_{\mathring{x}}^n\Big) = \nu\eta_{\bar{x}xt}^n. \tag{2.2}$$

This implicit scheme has the second order of spatial approximation and the first-order approximation in time. An example of a scheme of the second-order approximation is the following scheme considered in [4]:

$$\eta_t^n + \frac{1}{2}(c_0 + a\eta^{n+1})\eta_{\mathring{x}}^{n+1} + \frac{1}{2}(c_0 + a\eta^n)\eta_{\mathring{x}}^n = \nu\eta_{\bar{x}xt}^n \tag{2.3}$$

which is an analogue of the well-known Crank–Nicolson scheme widely used for solution of evolutionary equations. This scheme is nonlinear and requires iterations over nonlinearity in its implementation. Its modification was applied for the numerical implementation of the Nwogu model [18].

Harmonic analysis shows that difference schemes (2.2) and (2.3) are unconditionally stable in linear approximation because the transition factor $\rho$ of these schemes satisfies the equality $|\rho| \equiv 1$. If we neglect the dispersive summand in equation (2.1), then difference schemes (2.2) and (2.3) with the coefficient $\nu = 0$ now approximate the dispersion-free equation

$$\eta_t + c_0\eta_x + a\eta\eta_x = 0 \tag{2.4}$$

and are also unconditionally stable. These two examples show that difference schemes for equations with dispersion can inherit stability properties of known schemes for dispersion-free equations. However, such continuity is not always the case. Let us present the corresponding example.

In the scheme

$$\eta_t^n + (c_0 + a\eta^n)\eta_{\mathring{x}}^n = \nu\eta_{\bar{x}xt}^n \tag{2.5}$$

considered in [4] the convective terms are approximated by central differences from the $n$th layer and functions from the $(n+1)$th time layer are used only in terms containing time derivatives. An analogue of such scheme forms the base of algorithms of one splitting method for a system of NLD equations [10, 16].

Scheme (2.5) is weakly stable in linear approximation because its transition factor satisfies the inequalities

$$1 \leqslant \max_{\xi\in[0,\ \pi]} \left(|\rho|^2\right) \leqslant 1 + \frac{c_0^2}{4\nu}\,\tau^2$$

where $\xi = k\Delta x$, $k$ is the wave number of a harmonic. If we take $\nu = 0$ in scheme (2.5), then for $æ = \tau/\Delta x = \mathrm{const}$ it becomes unconditionally unstable. Thus, the consideration of a dispersive summand improves the stability of the scheme and transfer it to the class of weakly stable schemes. This fact established by the authors for scheme (2.5) takes place in the general case too. Namely, as a rule, schemes for the dispersion-free shallow water model are stable under more strict conditions than similar schemes for NLD equations.

As an example we consider a finite difference scheme with recalculation for system of nonlinear equations (1.8) proposed in [19]. In order to describe the properties of this scheme not detected by other authors, we write its linear analogue obtained by approximation of system (1.9), i.e.,

$$\begin{aligned} &\frac{\eta^* - \eta^n}{\tau} + h_0 u^n_{\mathring{x}} = 0 \\ &u^n_t + g\Big(\frac{1}{2}\eta^*_{\mathring{x}} + \frac{1}{2}\eta^n_{\mathring{x}}\Big) = \nu u^n_{\bar{x}xt} \\ &\eta^n_t + h_0\Big(\frac{1}{2}u^{n+1}_{\mathring{x}} + \frac{1}{2}u^n_{\mathring{x}}\Big) = 0 \end{aligned} \tag{2.6}$$

where $\nu = h_0^2/3$. In this scheme first we calculate the auxiliary value $\eta^*$ by explicit formulas and then obtain the solution $u^{n+1}$ to the difference motion equation by the scalar sweep method, and after that recalculate the free boundary $\eta^{n+1}$ on the new time layer by explicit formulas.

Excluding the value $\eta^*$ from the finite difference motion equation, we come to the system of two difference equations

$$\eta^n_t + \frac{h_0}{2}\Big(u^{n+1}_{\mathring{x}} + u^n_{\mathring{x}}\Big) = 0, \qquad u^n_t + g\eta^n_{\mathring{x}} = \frac{\tau c_0^2}{2} u^n_{\mathring{x}\mathring{x}} + \nu u^n_{\bar{x}xt} \tag{2.7}$$

where

$$u^n_{\mathring{x}\mathring{x},j} = \frac{u^n_{\mathring{x},j+1} - u^n_{\mathring{x},j-1}}{2\Delta x}.$$

Motion equation (2.7) shows that in the linear case the recalculation procedure is equivalent to introducing an artificial viscosity into the scheme.

Harmonic analysis gives the following stability condition:

$$c_0 æ \leqslant 1 + \sqrt{1 + \frac{4}{3}\delta^2} \tag{2.8}$$

where $\delta$ is the parameter characterizing the degree of the grid resolution relative to the typical depth $h_0$:

$$\delta = \frac{h_0}{\Delta x}. \tag{2.9}$$

For sufficiently fine grids stability condition (2.8) can be replaced by the following restriction on the time step:

$$\tau \leqslant \frac{2}{\sqrt{3}}\tau_0 \approx 1.15\tau_0$$

where $\tau_0 = h_0/c_0$ is the typical time necessary for a wave spreading with the velocity $c_0$ to travel the distance equal to the typical depth $h_0$. The more strict inequality $c_0 æ \leqslant 2$ was used previously as the stability condition for Peregrine scheme (2.6), which is the stability condition for a scheme approximating dispersion-free linear shallow water equations (1.13) for $\nu = 0$.

We note that finite difference schemes considered above are implicit. Their implementation generally uses the scalar sweep method, which may require additional restrictions on grid steps to ensure well definiteness of the algorithm. It is also worth noting that not all implicit schemes for systems of NLD equations are unconditionally stable. For example, if we do not apply recalculations in finite difference Peregrine scheme (2.6), i.e., use scheme (2.7) with the excluded viscous term $0.5\tau c_0^2 u^n_{\mathring{x}\mathring{x}}$, then it becomes only weakly stable under the condition

$$c_0 æ \leqslant 2\left(1 + \sqrt{1 + \frac{4}{3}\delta^2}\right).$$

**Remark 2.1.** In almost all papers related to numerical modelling within the framework of NLD models the problems of stability of difference schemes either were not discussed at all (referring to the fact that only implicit schemes are applied), or such restrictions on the Courant number were given that cause the necessity of refining the time step in a certain proportion to refining the spatial mesh size.

At the same time, in the simulation of the spread of undular bores, which is possible only within the NLD theory, an acceptable description of dispersive effects in a shallow water area is obtained for $\delta \in [2,\ 4]$ (according to the results of [12]). A similar result was obtained in [11], namely, it was shown that in solution of practical problems of long spread of waves in the ocean with subsequent going to the shelf, the choice of the spatial mesh size must correspond to the value $\delta = 1$ on the shallow area and to $\delta = 4$ in the deep part of the ocean. In this case stability conditions of form (2.8) mean that for a finite value of the parameter $\delta$ the calculations based on equations with dispersion can be performed with a time step $\tau$ increased in comparison to similar schemes for dispersion-free equations.

# 3 Analysis of dispersive properties of finite difference methods

Analysis of approximation errors shows that in schemes (2.2), (2.3), (2.5) considered above the 'scheme' dispersion has the same form as the dispersion of the approximated equation, and for finite grid steps this may cause distortions in the dispersive flow pattern described by the NLD model.

Let us consider the case of scheme (2.2) in detail and calculate the principal term of its phase error $\Delta\varphi$ for $æ = \mathrm{const}$ [23]. We have

$$\Delta\varphi = -\frac{c_0 æ}{6}\xi^3 - \frac{c_0^3 æ^3}{12}\xi^3 + O(\mu^4).$$

It is easy to check that the phase error has the same order on $\xi$ as the 'physical' dispersion of the model. Therefore, correct description of dispersive properties of the model by a difference scheme requires the following restriction:

$$\frac{c_0 æ}{6}\xi^3 + \frac{c_0^3 æ^3}{12}\xi^3 \ll c_0 æ \frac{\nu}{\Delta x^2}\xi^3$$

which implies the inequality

$$c_0 æ \ll \sqrt{2(\delta^2 - 1)}. \tag{3.1}$$

Relation (3.1) implies that the spatial step $\Delta x$ must be less than the depth $h_0$. In this case for sufficiently fine grids ($\delta \gg 1$) the scheme dispersion cannot damp the dispersion of the model. Thus, for some schemes approximating equations with dispersion, for example, for scheme (2.2) considered here, a good approximation of the 'physical' dispersion can be obtained only on fine calculation grids. However, one can indicate finite difference schemes for NLD equations where the 'scheme' dispersion does not dump the 'physical' one even on relatively coarse grids. For example, in Peregrine scheme (2.7) the principal part of the phase error is calculated by the formula

$$\Delta\varphi = \frac{c_0 æ}{24}\left(c_0^2 æ^2 - 4\right)\xi^3 + O(\xi^4).$$

It is seen that if $c_0 æ = 2$, then the additional dispersion introduced by difference scheme (2.7) is minimal and the scheme remains stable.

The above analysis of approximation errors has shown that in some finite difference schemes the 'scheme' dispersion may dump the dispersion of NLD model. To get rid of such undesirable property, a series of papers proposed to modify the scheme by introducing principal terms of the phase error with the opposite sign into it. This allows us to cancel the main by approximation order dispersion terms of scheme origin and to improve the representation of the dispersion in the approximated model.

Such technique of increasing the accuracy of numerical calculations was used in widely cited papers [1, 18] and others. For example, the main result of [18] consisted in derivation of an NLD model with enhanced representation of the dispersion of a full hydrodynamic model, the approximation of the NLD model was performed with the use of the predictor–corrector method where main calculations applied a Crank–Nicolson scheme and then the obtained values were used to calculate the term of 'scheme' dispersion, which was included into calculations on the correction step as a known right-hand side of equations. The analysis of stability was performed in this paper for the scheme approximating linear equations (1.12) and the right-hand side mentioned above as well as the correction step were ignored. The Neumann analysis leads to an expression for eigenvalues of the transition matrix whose analytic study is rather complicated in the general case. As the result, the authors present a rough stability condition in the form of the inequality $c_0 æ < 2$, i.e., in essence, the stability condition for an analogue of a difference scheme for linear shallow water equations (1.13). However, assuming $\beta = -1/3$, we get a particular case of a difference scheme approximating the Peregrine NLD equations, and as has been shown by our analysis, the eigenvalues of its transition matrix does not exceed one under more weak condition (2.8) discussed in the previous section.

Later, a difference scheme of the fourth-order approximation was applied to the Nwogu model and its completely nonlinear variant. In this case the 'scheme dispersion' introduced by the third-order derivatives is excluded automatically. A finite difference predictor-corrector scheme was developed in [25]; this scheme was based on the known Adams–Bashforth method of the third-order approximation (predictor step) and Adams–Moulton method of the fourth-order approximation (corrector step). Despite the popularity of the proposed method, the stability of this difference scheme was not discussed in the papers where it was applied. The time step was usually taken in calculations proportionally to the spatial mesh size with some experimentally chosen coefficient of proportionality.

## 4 Special construction techniques for numerical algorithms

The method of construction of difference schemes by direct approximation of all terms of equations including mixed third-order derivatives has many drawbacks. A method allowing one to split NLD equations and reduce the implementation of the corresponding numerical algorithm to successive solution of an ODE system and an elliptic equation seems to be more efficient. Such technique was suggested in [3, 4] where the equation $u_t + u_x + uu_x = u_{xxt}$ was written in the form $(u - u_{xx})_t + u_x + uu_x = 0$ convenient for splitting into two equations

$$Q_t + u_x + uu_x = 0, \qquad u - u_{xx} = Q \tag{4.1}$$

which implies difference approximations according to the form of these equations.

We extend this technique to basic model (1.2) rewriting it in the following divergent form:

$$H_t + (HU)_x = 0, \quad (HU)_t + (HuU)_x + (HJu)_x + p_x = p_0 h_x \tag{4.2}$$

where $U = u + J$. Further we transform the part of the equation related to the pressure. Using expressions (1.3), separate the following terms from $p_x - p_0 h_x$:

$$-\Big(\frac{H^3}{3}u_{xt}\Big)_x + \frac{H^2}{2}u_{xt}h_x = -\Big(\frac{H^3}{3}u_x\Big)_{xt} + (H^2 H_t u_x)_x + \Big(\frac{H^2}{2}u_x h_x\Big)_t - \Big(\frac{H^2}{2}h_x\Big)_t u_x = Q_t + Q_2$$

and denote the remaining terms by $Q_1$. In this case,

$$Q = -\Big(\frac{H^3}{3}u_x\Big)_x + \frac{H^2}{2}u_x h_x, \quad Q_2 = (H^2 H_t u_x)_x - \Big(\frac{H^2}{2}h_x\Big)_t u_x.$$

Now we can rewrite the motion equation from (4.2) by analogy with (4.1), i.e.,

$$V_t + (HuU)_x + (HJu)_x = -Q_1 - Q_2, \qquad HU + Q = V. \tag{4.3}$$

One of the most widely used methods of construction of numerical algorithms consists in the following. First we calculate $H^{n+1}$ from the continuity equation using an explicit (or implicit relative to $H$) scheme. After that, based on an appropriate difference method approximating the first equation of (4.3), we explicitly calculate $V^{n+1}$ from the obtained values of $H^{n+1}$. Then we solve the elliptic equation $HU + Q = V^{n+1}$ (in the one-dimensional case this is a second-order ordinary differential equation) with respect to $u^{n+1}$ (using the sweep method in the one dimensional case).

In [16] and other papers of that author a similar approach was used for construction of finite difference schemes for a modified Green–Naghdi model and Aleshkov's model written in a free nonconservative form.

For $J = 0$ some variants of finite difference schemes with separation of $Q_t$ were also considered in [10, 22] where some modifications were proposed, namely, the use of staggered grids and finite difference schemes with weights. In the case $J \neq 0$ this approach was used in [25].

Two other approaches, which can be illustrated on the basic model, consists in the separation of spatial and time derivatives with the use of additional variables $\varphi = u_t$ and $\psi = \eta_t$ and thus we get elliptic equations for determination of $\varphi$ and $\psi$ and an ODE system for $u$ and $\eta$. In the case $J = 0$ such splitting requires only one variable $\varphi = u_t$. For numerical implementation of NLD models admitting the Galilean transformation, the introduction of the variable $d = u_t + uu_x$ is rather efficient. These variants were considered in [10, 15] where some finite difference methods were constructed on their base for the Zheleznyak–Pelinovsky model and some other NLD models.

Another variant of splitting leads to an extended system of equations consisting of an elliptic equation for the dispersive component of the pressure $p$ integrated over depth from (1.3) and a hyperbolic system of equations with a right-hand side. Such splitting preserving the continuity of numerical algorithms developed for shallow water equations proved to be fruitful for construction of numerical solution algorithms for a flat [14] and spherical [13] geometry.

## 5 Conclusions

This paper indicates that the study of finite difference methods applied to solution of hydrodynamics NLD equations is behind the pace of development of NLD models themselves and their practical application. On the one hand, the edge of numerical modelling of complex problems describing wave modes is formed by such problems as the analysis of instability related to nonlinearity of equations and consideration of sharp changes in the bottom surface [17], an enhanced description of the dispersion of ‘short’ waves, and on the other hand, there are many gaps in the study of fundamental properties of difference methods even in the case of the simplest NLD equations. The application of a hierarchical approach allows one to ‘get rid’ of those gaps in existing methods of numerical modelling, to clarify their nature, and reveal the difference from the solution methods for dispersion-free shallow water equations. In the future such studies will allow us to build a hierarchy according to properties determining the accuracy and operability of numerical algorithms and obtain their relative performance on efficiency [9].

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